the reference time. A thermal and mechanical shock will be applied to the left end of the beam. The thermal shock is assumed to be produced by a sudden change in the ambient temperature. This sudden temperature change is transformed to the beam by convection, resulting in a large thermal gradient at the exposed end. This case of loading is a realistic model of a shock wave formation. The variation of the temperature and axial stress along the beam is plotted for different times in Figs. 1 and 2. The same type of loading is imposed for a cantilever beam and the curves for the temperature and axial stress variation vs time are shown in Figs. 3 and 4. The main difference between the results is the value of the stress at the clamped vs free edges for two different types of beams. For the free-free beam, the axial stress at the end where the load is imposed varies, while at the opposite edge this variation of the axial stress is near zero. On the other hand, for the clamped beam where its free edge is exposed to the shock, the clamped edge experiences high variations of the axial stress. The case is quite relevant and shows the validity and power of the proposed method.

Conclusion

Problems of coupled thermoelasticity, while having a wide range of engineering applications, have not received a great deal of attention. The main reasons are complicated analytical solutions and the lack of a sound numerical method. Bahar and Hetnarski[5,6] have recently published a series of papers proposing the state-space approach. That method is, of course, sound for one-dimensional problems, but application to two- and three-dimensional problems seems to have some difficiencies. The method proposed in this Note, which is based on the Galerkin method, does not encounter any major deficiencies in its application to two- and three-dimensional problems and thus at present, seems to be the most appropriate finite-element formulation for this class of problems.

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Nonlinear Free Vibration of Elastic Plates

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Introduction

THE effect of large deflection on the free nonlinear vibrations of elastic plates, based on von Kármán's large deflection plate theory, has been considered by many researchers. By applying Herrmann's theory, Chu and Herrmann[1] studied the influence of large deflections on the free vibrations of a rectangular plate with hinged immovable edges. By using the first-order approximation to that of Herrmann, Yamaki[2] investigated one-term solutions for simply supported and clamped plates for free and forced sinusoidal vibrations. Using the perturbation method of Poincare's as modified by Lighthill, Bauer[3] investigated the flexural vibrations of plates with simply supported and clamped boundary conditions, subjected to the step function and the exponentially decaying pulse. The influence of initial membrane stress on the free and forced nonlinear vibration of beams and rectangular plates was investigated by Eisley[4] by means of a simple extension of the results for the unstressed case by using single-mode representation.

In this paper, the influence of large amplitudes on free vibrations of square and circular plates for simply supported and clamped-in boundary conditions and for immovably constrained and stress-free edge conditions are studied by applying Anderson's[5,6] ultraspherical polynomial approximation (UPA) technique for solving the nonlinear differential equation in time. The results of the UPA technique are compared with the elliptic function method (EFM).

Square Plates

The dynamic von Kármán equations for the free vibration of plates may be written as

$$\nabla^4 F = E\left[\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2}\right)\right] \tag{1}$$

$$L(W,F) = D\nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} - h\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y}\right] = 0 \tag{2}$$

In these equations, the effects of both the longitudinal and rotatory inertia forces have been neglected and correspond to the first approximation to those of Herrmann as stated by Chu and Herrmann.[1] The boundary conditions considered are the simply supported (SS) and the clamped-in (CI); and the edge conditions considered are the stress-free (SF) and the immovably constrained (IC).[3] The solution for a simply supported square plate is assumed to be

$$W(x,y,t) = hf(t)\cos(\pi x/a)\cos(\pi y/a)$$

Airy's stress function is expressed as

$$F(x,y,t) = F^*(x,y)f^2(t)$$

Following Bauer's[3] procedure, we get, for the SF case,

$$\ddot{f}(t) + \omega^2 f(t) + \epsilon\omega^2 f^3(t) = 0 \tag{3}$$

where

$$\omega^2 = \frac{\pi^4 E h^2}{3\rho(1-\nu^2)a^4}$$

$$\epsilon = 0.19476(1-\nu^2)$$

A similar procedure is followed in other cases, and the resulting equation is of the form of Eq. (3). The values ω^2 and ϵ in each case are given in Table 1.

Circular Plates

The dynamic von Kármán equations for the free vibration of circular plates are written as

$$\nabla^4 F = -\frac{E}{r}\frac{\partial W}{\partial r}\frac{\partial^2 W}{\partial r^2} \tag{4}$$

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Table 1 Values of ϵ and ω^2 for different cases

Cases	Edge conditions	ϵ	Assumed solution	ω^2
Simply supported square plate	IC	$\frac{3}{8}(3-\nu)(1-\nu)$	$W(x,y,t)=hf(t)\cos(\pi x/a)$	$\frac{\pi^4 Eh^2}{3\rho(1-\nu^2)a^4}$
	SF	$0.19476\,(1-\nu^2)$	$\cos(\pi y/a)$	
Clamped-in square plate	IC	$\frac{27}{128}(1+\nu)+\frac{8.49}{32}(1-\nu^2)$	$W(x,y,t)=hf(t)\cos 2(\pi x/a)$	$\frac{32\pi^4 Eh^2}{27\rho(1-\nu^2)a^4}$
	SF	$0.22355\,(1-\nu^2)$	$\cos^2(\pi y/a)$	
Simply supported circular plate	IC	$\frac{(C_2+C_3)}{C_1}\frac{Eh^3}{32D}$	$W(r,t)=hf(t)\left[1-\frac{2(\nu+3)}{(\nu+5)}\right.$	$\frac{32DC_1}{\rho a^4 h}$
	SF	$\frac{(C_2+C_4)}{C_1}\frac{Eh^3}{32D}$	$\left.(r/a)^2+\frac{(1+\nu)}{(\nu+5)}(r/a)^4\right]$	
Clamped-in circular plate	IC	$\frac{(23-9\nu)(1+\nu)}{56}$	$W(r,t)=hf(t)$	$\frac{320D}{3\rho a^4 h}$
	SF	$\frac{9(1-\nu^2)}{56}$	$[1-2(r/a)^2+(r/a)^4]$	

Table 2 The influence of large amplitudes on the free vibration of square plates

Amplitude, f_m	SS-IC UPA	SS-IC EFM	SS-SF UPA	SS-SF EFM	CI-IC UPA	CI-IC EFM	CI-SF UPA	CI-SF EFM
	Ratio of the nonlinear period to the linear period, $\bar{\omega}$							
0.0	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999
0.1	0.9964	0.9989	0.9991	0.9991	0.9974	0.9974	0.9990	0.9989
0.2	0.9859	0.9886	0.9964	0.9965	0.9897	0.9923	0.9959	0.9959
0.3	0.9688	0.9769	0.9920	0.9921	0.9771	0.9825	0.9908	0.9909
0.4	0.9459	0.9599	0.9859	0.9885	0.9600	0.9684	0.9838	0.9865
0.5	0.9179	0.9397	0.9781	0.9835	0.9389	0.9532	0.9750	0.9804
0.6	0.8859	0.9162	0.9688	0.9769	0.9143	0.9351	0.9644	0.9726
0.7	0.8508	0.8917	0.9580	0.9690	0.8869	0.9171	0.9521	0.9633
0.8	0.8137	0.8635	0.9459	0.9599	0.8572	0.8950	0.9383	0.9527
0.9	0.7753	0.8371	0.9324	0.9496	0.8259	0.8718	0.9232	0.9434
1.0	0.7365	0.8082	0.9179	0.9384	0.7935	0.8505	0.9069	0.9306

Table 3 The influence of large amplitudes on the free vibration of circular plates

Amplitude, f_m	SS-IC UPA	SS-IC EFM	SS-SF UPA	SS-SF EFM	CI-IC UPA	CI-IC EFM	CI-SF UPA	CI-SF EFM
	Ratio of the nonlinear period to the linear period, $\bar{\omega}$							
0.0	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999
0.1	0.9992	0.9992	0.9987	0.9987	0.9976	0.9976	0.9993	0.9992
0.2	0.9968	0.9968	0.9948	0.9948	0.9906	0.9931	0.9971	0.9970
0.3	0.9928	0.9929	0.9884	0.9910	0.9790	0.9843	0.9934	0.9934
0.4	0.9873	0.9900	0.9795	0.9848	0.9633	0.9716	0.9883	0.9909
0.5	0.9804	0.9856	0.9684	0.9765	0.9439	0.9579	0.9819	0.9846
0.6	0.9720	0.9775	0.9551	0.9662	0.9211	0.9414	0.9741	0.9795
0.7	0.9622	0.9705	0.9399	0.9542	0.8956	0.9227	0.9651	0.9733
0.8	0.9512	0.9625	0.9229	0.9431	0.8679	0.9022	0.9549	0.9660
0.9	0.9390	0.9534	0.9044	0.9283	0.8384	0.8805	0.9436	0.9577
1.0	0.9258	0.9433	0.8845	0.9149	0.8078	0.8607	0.9312	0.9485

and

$$L'(W,F)=D\nabla^4 W+\rho h\frac{\partial^2 W}{\partial t^2}-\frac{h}{r}\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\frac{\partial W}{\partial r}\right)=0 \tag{5}$$

In these equations, axial symmetry is assumed and the effect of rotatory inertia is neglected. The boundary and edge conditions considered are as in the square plates. Following a similar procedure, the resulting equation is of the form of Eq. (3), and the corresponding values of ω^2 and ϵ in each case are given in Table 1.

Nonlinear Period of Vibration

The initial conditions are $f(t)=f_m$ and $\dot{f}(t)=0$ at $t=0$. Let

$$f=F_1 \cos\psi \tag{6}$$

where $\psi=\omega t+\theta$ be the solution to Eq. (3). Following Anderson[5,6] yields

$$\dot{F}_1=\epsilon\omega F_1^3\left[\frac{\sin 4\psi}{8}+\frac{\sin 2\psi}{4}\right]$$

$$\dot{\theta}=\epsilon\omega F_1^2\left[\frac{3}{8}+\frac{\cos 4\psi}{8}+\frac{\cos 2\psi}{2}\right] \tag{7}$$

Expanding the right-hand sides of Eq. (7) in ultraspherical polynomials and after integration yields

$$F_1=F_0$$

and

$$\theta=\epsilon\omega F_1^2\left[\frac{3}{8}+\frac{C_4}{8}+\frac{C_2}{2}\right]t+\theta_0$$

where $F_0=F_1|_{t=0}$ and $\theta_0=\theta|_{t=0}$ are constants of integration determined from the initial conditions:

$$C_n=\frac{\Gamma(\lambda+1)J_\lambda(n\pi)}{(n\pi/2)^\lambda}, \quad n=1,2,3,4$$

where λ = ultraspherical polynomial index, $\Gamma(\ )$ = gamma function, and $J_\lambda(\ )$ = Bessel function. The ratio of the nonlinear period to the linear period when $\lambda=0$ is

$$\bar{\omega}=\frac{1}{(1+0.50475\epsilon f_m^2)}$$

The influence of large amplitudes on the free vibration of square and circular plates for different boundary and edge conditions is tabulated in Tables 2 and 3, respectively, and compared with that obtained by the elliptic function method.

Conclusion

As shown in Tables 2 and 3, the results obtained by the ultraspherical polynomial approximation technique agree well with the results obtained by the elliptic function method, and it is seen that in each case, the period decreases with increasing amplitude.

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